Correlation Risk in CDS Pricing

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Initial Version: August 16, 2016
Current Version: November 10, 2017

Abstract

We propose a hybrid rates-credit model for pricing credit default swaps (CDS) in the presence of a correlation between rates and credit. We suppose a beta-blend model for rates and a Black-Karasinski model for credit, so allowing some interest rate skew to be modelled and ensuring positive default intensities. Using the perturbation expansion techniques proposed by Horvath et al. (2017) we deduce an analytic expression for the relevant Green's function operator which is asymptotically valid in the limit of low rates and intensities. We use this to produce approximate expressions for CDS prices. It is also shown how, converesly, these results can be used through a bootstrapping procedure to calibrate the model to a term structure of CDS rates. We consider also the pricing of defaultable Libor payments, which capability would be required for CDS with floating rate coupons, or else for interest rate swap extinguishers.

1 Introduction

We consider reduced form modelling of credit derivative prices where both the interest rate and the credit intensity (or spread) are considered to be stochastic and potentially correlated. We base our approach on that pioneered by Schönbucher (1999) who took both processes to be normal mean-reverting diffusions, in other words governed by the gaussian short rate model of Hull and White (1990). Solutions were found by constructing a two-dimensional tree. As was pointed out by Schönbucher (1999), it is a straightforward matter to extend his model to non-gaussian processes. A number of authors have followed this suggestion taking the credit process to be lognormal, governed by a Black and Karasinski (1991) short rate model which, although less tractable than a gaussian model, ensures that credit spreads stay positive (and thus that survival probabilities are decreasing functions of time). Jobst and Zenios (2001) sought to price portfolios of bonds, modelling the credit spread for securities in a given rating class in this way, coupled with a Hull-White interest rate model, but also allowing rating class migrations to take place. A similar approach with only rates and credit default risk was used by Cortina (2007) to provide analytic solutions for the prices of defaultable bonds in the assumed absence of correlation, and by Pan and Singleton (2007) who considered the joint distribution of credit spreads and default loss rates implied by CDS market data.

We propose here a rates-credit hybrid model for CDS pricing. Interest rates are taken to be governed by the beta-blend short rate model introduced by Horvath et al. (2017): this is a one-parameter family of models where the choice of the parameter β allows the skewness to be fitted to the term structure implied by market prices of interest rate options. It provides an interpolation between the popular Hull-White ($\beta \uparrow 1$) and Black-Karasinski ($\beta = 0$) models. The credit model we assume to be Black-Karasinski for the reasons already given. The two models we allow to be correlated, the impact of this correlation being our main interest in the present work. Horvath et al. (2017) has shown how these models can be solved analytically to provide Green's functions which are asymptotically accurate in the limit of low rates. It is a straightforward matter to use these Green's functions to calculate conditional values of future cash flows (in the interest rate case); likewise, in the case of credit modelling, we are able to obtain conditional survival probabilities and, by the same token, conditional default probability distributions. Our interest in the present work is to create

a bivariate Green's function where both rates and credit are stochastic and possibly correlated. Typically it is only possible to price credit derivatives such as CDS in such circumstances numerically, but we show how, armed with an asymptotically valid representation of the requisite Green's function, CDS prices can be inferred to good accuracy by analytic means from known credit spreads (or vice versa) in the presence of non-zero correlation.

The layout of the paper is as follows. We set out our stochastic modelling assumptions and associated no-arbitrage conditions then derive the associated PDE in §2. We introduce in §3 an asymptotic scaling with the assumption that both interest rates and credit spreads are small, on which basis we are able to derive a Green's function solution for our PDE as an asymptotic expansion. Calibration of the model to satisfy the specified no-arbitrage conditions is considered in §4. We use our Green's function in §5 to derive the price of the protection leg of a CDS. It is shown in §6 how the (fixed) coupon leg is priced, taking into account the possible impact of accrued interest if this is required to be paid upon default. The reverse task of calibrating the model to CDS prices, if these rather than risky bond prices are considered to be known *a priori*, is considered in §7. We go on to show in §8 how defaultable Libor payments can be priced, which facility is required where a CDS (or credit linked note, or other credit-contingent instrument) pays floating rate coupons. We again take account of accrued interest payments at default. Finally, comparisons are made in §9 of results obtained using our formulae with more accurate finite difference results. They are seen to be highly favourable.

2 Modelling Assumptions

We model the interest short rate and the instantaneous credit default intensity as stochastic processes. These are taken, respectively, to be defined by a beta-blend model (see Horvath et al., 2017) and the lognormal model of Black and Karasinski (1991). We shall find it convenient to work with auxiliary variables $\hat{x}_t$ and $\hat{y}_t$ satisfying the following Ornstein-Uhlenbeck processes:

$$d\hat{x}_t = -\alpha_r \hat{x}_t dt + \sigma_r(t)\, dW_t^1, \tag{1}$$

$$d\hat{y}_t = -\alpha_\lambda \hat{y}_t dt + \sigma_\lambda(t)\, dW_t^2, \tag{2}$$

where dW_t^1 and dW_t^2 are correlated Brownian motions under the risk-neutral measure with

$$\operatorname{corr}(W_t^1, W_t^2) = \rho_{r\lambda}.$$

These auxiliary variables are related to the interest short rate r_t and the credit default intensity λ_t, respectively, by

$$(1-\beta)\, r_t + \beta\overline{r}(t) = (\overline{r}(t) + (1-\beta) r^*(t))\mathcal{E}\left(\frac{(1-\beta)\hat{x}_t}{|\overline{r}(t)|^\beta}\right), \tag{3}$$

$$\lambda_t = (\overline{\lambda}(t) + \lambda^*(t))\mathcal{E}(\hat{y}_t), \tag{4}$$

with $\beta \in [0,1)$ assumed. Here $\overline{r}(t)$ is the instantaneous forward rate, $\overline{\lambda}(t)$ the associated credit spread (see (8) below) and $\mathcal{E}(X_t) := \exp\left(X_t - \frac{1}{2}[X]_t\right)$ is a stochastic exponential with $[X]_t$ the quadratic variation of a process X_t.[1] The required form of the configurable functions $r^*(t)$ and $\lambda^*(t)$ is determined by calibration of the model to satisfy the no-arbitrage conditions set out below. We further assume that $\hat{x}_0 = \hat{y}_0 = 0$, with $t = 0$ the "as of" date for which the model is calibrated. As can be seen, the beta-blend model for the interest rate represents a hybrid between the Hull-White model ($\beta \uparrow 1$) and the Black-Karasinski model ($\beta = 0$). The credit intensity model is taken to be Black-Karasinski.

[1] We observe that, for $\beta \in (0,1)$, r_t is not defined at times t for which $\overline{r}(t) = 0$. This problem can be mitigated by replacing $|\overline{r}(t)|^\beta$ on the denominator in (3) with $(|\overline{r}(t)|+(1-\beta)\delta)^\beta$ for some small smoothing parameter $\delta > 0$. This will ensure a positive denominator as well as maintaining Hull-White and Black-Karasinski as the limiting case models when $\beta \to 1$ and $\beta \to 0$ respectively. The analysis below goes through unaffected making this substitution throughout.

The no-arbitrage condition

The formal no-arbitrage constraints which determine the functions $r^*(t)$ and $\lambda^*(t)$ are as follows:

$$E\left[e^{-\int_0^t r_s\,ds}\right] = D(0,t), \tag{5}$$

$$E\left[e^{-\int_0^t (r_s+\lambda_s)\,ds}\right] = B(0,t) \tag{6}$$

under the martingale measure for $0 < t \leq T_m$, where T_m is the longest maturity date for which the model is calibrated,

$$D(t_1,t_2) = e^{-\int_{t_1}^{t_2} \overline{r}(s)\,ds} \tag{7}$$

is the t_1-forward price of the t_2-maturity zero coupon bond and

$$B(t_1,t_2) = e^{-\int_{t_1}^{t_2} (\overline{r}(s)+\overline{\lambda}(s))\,ds} \tag{8}$$

the corresponding risky bond price. We shall assume the bond prices can be ascertained at the initial time $t = 0$ from the market, whence we can view (7) and (8) as *defining* the forward rate $\overline{r}(t)$ and associated credit spread $\overline{\lambda}(t)$, respectively.

Derivation of governing PDE

We consider the general problem of pricing a cash security with maturity T whose payoff depends on $\hat{x}_T$ and $\hat{y}_T$. We will also look in §5 at the case of a protection leg whose payoff depends on $\hat{x}_{\tau^*}$ and $\hat{y}_{\tau^*}$, where τ^* is a stopping time in $(0,T]$. We introduced the convenient shorthand notation that, for a process X_t and deterministic function $f(.,.)$,

$$\mathcal{E}_x(f(X_t,t)) := \mathcal{E}(f(X_t,t))|_{X_t=x},$$

in terms of which we can re-write (3) and (4) as $r_t = r(\hat{x}_t,t)$ and $\lambda_t = \lambda(\hat{y}_t,t)$, where

$$r(\hat{x},t) := \frac{1}{1-\beta}\left((\overline{r}(t)+(1-\beta)r^*(t))\mathcal{E}_{\hat{x}}\left(\frac{(1-\beta)\hat{x}_t}{|\overline{r}(t)|^\beta}\right) - \beta\overline{r}(t)\right), \tag{9}$$

$$\lambda(\hat{y},t) := (\overline{\lambda}(t)+\lambda^*(t))\mathcal{E}_{\hat{y}}(\hat{y}_t), \tag{10}$$

Writing the price of the security at time $t \in [0,T]$ as $f_t^T = \hat{f}(\hat{x}_t,\hat{y}_t,t)$, we can infer by application of the Feynman-Kac theorem to (1) and (2) in the standard manner that the function $\hat{f}(\hat{x},\hat{y},t)$ satisfies the following backward diffusion equation:

$$\left(\frac{\partial}{\partial t} + \hat{\mathcal{L}} - r(\hat{x},t) - \lambda(\hat{y},t)\right)\hat{f}(\hat{x},\hat{y},t) = 0, \tag{11}$$

where

$$\hat{\mathcal{L}} := -\alpha_r\hat{x}\frac{\partial}{\partial\hat{x}} - \alpha_\lambda\hat{y}\frac{\partial}{\partial\hat{y}} + \frac{1}{2}\left(\sigma_r^2(t)\frac{\partial^2}{\partial\hat{x}^2} + 2\rho_{r\lambda}\sigma_r(t)\sigma_\lambda(t)\frac{\partial^2}{\partial\hat{x}\partial\hat{y}} + \sigma_\lambda^2(t)\frac{\partial^2}{\partial\hat{y}^2}\right) \tag{12}$$

with in general $f_T^T = \hat{P}(\hat{x}_T,\hat{y}_T)$ for some payoff function $\hat{P}(.)$.[2] In the absence of closed form solutions to (11) and guided by the work of Hagan et al. (2005) and Horvath et al. (2017), we propose a perturbation expansion approach as follows.

[2] In the following we will not in fact consider payoffs dependent on $\hat{y}$, although the methods employed could easily be extended to take account of such an eventuality.

3 Perturbation Expansion

3.1 Definition of Scaled Variables

For both short rate models we apply a "low rates" assumption. To this end we define, taking T_m to be the longest time to maturity for which the model is calibrated, small parameters

$$\epsilon_r := \frac{1}{\alpha_r T_m} \int_0^{T_m} \overline{r}(t)\, dt \tag{13}$$

$$\epsilon_\lambda := \frac{1}{\alpha_\lambda T_m} \int_0^{T_m} \overline{\lambda}(t)\, dt \tag{14}$$

and $\mathcal{O}(1)$ functions

$$\begin{aligned} \tilde{r}(t) &:= \epsilon_r^{-1} \frac{\overline{r}(t)}{1-\beta}, \\ \tilde{r}^*(t) &;= \epsilon_r^{-1} r^*(t) \end{aligned} \tag{15}$$

and

$$\begin{aligned} \tilde{\lambda}(t) &= \epsilon_\lambda^{-1} \overline{\lambda}(t), \\ \tilde{\lambda}^*(t) &= \epsilon_\lambda^{-1} \lambda^*(t). \end{aligned} \tag{16}$$

We further define a new scaled variable x_t and associated scaled volatility $\sigma_x(t)$ by

$$\begin{aligned} x_t &:= \epsilon_r^{-\beta} \hat{x}_t e^{\alpha_r t}, \\ \sigma_x(t) &= \epsilon_r^{-\beta} \sigma_r(t) e^{\alpha_r t}. \end{aligned} \tag{17}$$

Likewise we define

$$\begin{aligned} y_t &:= \hat{y}_t e^{\alpha_\lambda t}, \\ \sigma_y(t) &:= \sigma_\lambda(t) e^{\alpha_\lambda t}. \end{aligned} \tag{18}$$

Here the exponential time scaling is to facilitate removal of the mean reverting drift terms in (12). We further define new functional forms $f(\cdot)$ and $P(\cdot)$ by:

$$\begin{aligned} f(x,y,t) &:= B(0,t) f_t^T|_{x_t = x, y_t = y}, \\ P(x_T, y_T) &\equiv \hat{P}(\hat{x}_T, \hat{y}_T), \end{aligned}$$

where $\hat{x}_T$ and $\hat{y}_T$ are related to x_T and y_T by (17) and (18), respectively. In this notation, (1) and (2) can be re-expressed as

$$dx_t = \sigma_x(t)\, dW_t^1, \tag{19}$$

$$dy_t = \sigma_y(t)\, dW_t^2 \tag{20}$$

and (11) as

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - \epsilon_r h(x,t) - \epsilon_\lambda g(y,t) \right) f(x,y,t) = 0 \tag{21}$$

where

$$\mathcal{L}[\cdot] := \frac{1}{2} \left(\sigma_x^2(t) \frac{\partial^2}{\partial x^2} + 2\rho_{r\lambda} \sigma_x(t) \sigma_y(t) \frac{\partial^2}{\partial x \partial y} + \sigma_y^2(t) \frac{\partial^2}{\partial y^2} \right), \tag{22}$$

$$h(x,t) := h(x,t,t) \tag{23}$$

$$g(y,t) := g(y,t,t) \tag{24}$$

with

$$h(x,t,t_1) := (\tilde{r}(t_1)+\tilde{r}^*(t_1))\mathcal{E}_x\left(F_\beta(t_1)x_t\right) - \tilde{r}(t_1), \quad t_1 \geq t, \tag{25}$$

$$F_\beta(t) := \frac{(1-\beta)^{1-\beta}e^{-\alpha_r t}}{|\tilde{r}(t)|^\beta}, \tag{26}$$

$$g(y,t,t_1) := (\tilde{\lambda}(t)+\tilde{\lambda}^*(t))\mathcal{E}_y\left(e^{-\alpha_\lambda t_1}y_t\right) - \tilde{\lambda}(t_1), \quad t_1 \geq t. \tag{27}$$

We seek a Green's function solution for (21) as a joint power series in ϵ_r and ϵ_λ.

3.2 Derivation of Green's Function

We follow the methodology propounded by Hagan et al. (2005) and subsequently developed by Pagliarani et al. (2011) in observing that the operator

$$\mathcal{U}(t;v) = \exp\int_t^v (\mathcal{L}(u) - \epsilon_r h(x,u) - \epsilon_\lambda g(y,u))du \tag{28}$$

is a formal solution of (21) for $t \leq v$ subject to $\mathcal{U}(v;v) = I$. The Green's function is then the integral kernel of $\mathcal{U}(t;v)$, viz.

$$G(x,y,t;\xi,\eta,v) = \mathcal{U}(t;v)(x,y;\xi,\eta). \tag{29}$$

Note that the corresponding Green's function for $f_t^T|_{x_t=x,y_t=y}$ will be given straightforwardly[3] by

$$G^*(x,y,t;\xi,\eta,v) = B(t,v)G(x,y,t;\xi,\eta,v). \tag{30}$$

Again following Hagan et al. (2005) we consider first the limiting problem with $\epsilon_r = \epsilon_\lambda = 0$. The required solution can be written

$$\mathcal{U}_{0,0}(t;v) := \exp\int_t^v \mathcal{L}(u)du. \tag{31}$$

We will in all cases be interested in so-called free-boundary Green's function solutions which tend to zero as $x, y \to \pm\infty$. The Green's function solution subject to these conditions is well known. It is given by:

$$G_{0,0}(x,y,t;\xi,\eta,v) = \frac{\partial^2}{\partial\xi\partial\eta}N_2(\xi - x, \eta - y; R(t,v)), \quad t \leq v \tag{32}$$

where $N_2(x,y;R(t,v))$ is a bivariate Gaussian probability distribution function with mean $\mathbf{0}$ and covariance matrix

$$R(t,v) := \begin{pmatrix} I_x(t,v) & I_\rho(t,v) \\ I_\rho(t,v) & I_y(t,v) \end{pmatrix} \tag{33}$$

with

$$I_x(t_1,t_2) := \int_{t_1}^{t_2} \sigma_x^2(u)\,du \tag{34}$$

$$I_y(t_1,t_2) := \int_{t_1}^{t_2} \sigma_y^2(u)\,du. \tag{35}$$

$$I_\rho(t_1,t_2) := \rho_{r\lambda}\int_{t_1}^{t_2} \sigma_x(u)\sigma_y(u)\,du. \tag{36}$$

[3]The required Green's function must be of the form $G^*(.) = CB(0,t)^{-1}G(.)$. Correct normalisation of the new Green's function at $t = v$ then requires $CB(0,v)^{-1} = 1$, from which we deduce (30).

Before deriving higher order terms in our proposed Green's function expansion, we introduce some additional notation. First, following Horvath et al. (2017), we propose that the form of the configurable function $\tilde{r}^*(t)$ required to satisfy (5) is

$$\tilde{r}^*(t) = \sum_{i=0}^{\infty} \epsilon_r^i r_i^*(t). \tag{37}$$

We further infer from the structure of (21) that the form of the expansion for $\tilde{\lambda}^*(t)$ required to satisfy (6) is

$$\tilde{\lambda}^*(t) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \epsilon_r^i \epsilon_\lambda^j \lambda_{i,j}^*(t). \tag{38}$$

On this basis we can now expand $h(x, t, t_1)$ as

$$h(x, t, t_1) = \sum_{i=0}^{\infty} \epsilon_r^i h_i(x, t, t_1) - \tilde{r}(t_1) \tag{39}$$

where

$$\begin{aligned} h_0(x, t, t_1) &:= (\tilde{r}(t_1) + r_0^*(t_1))\mathcal{E}_x(F_\beta(t_1)x_t) \\ h_i(x, t, t_1) &:= r_i^*(t_1)\mathcal{E}_x(F_\beta(t_1)x_t), \quad i > 0. \end{aligned}$$

and similarly

$$g(y, t, t_1) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \epsilon_r^i \epsilon_\lambda^j g_{i,j}(x, t, t_1) - \tilde{\lambda}(t_1) \tag{40}$$

where

$$\begin{aligned} g_{0,0}(y, t, t_1) &:= (\tilde{\lambda}(t_1) + \lambda_{0,0}^*(t_1))\mathcal{E}_y(e^{-\alpha_\lambda t_1} y_t) \\ g_{i,j}(y, t, t_1) &:= \lambda_{i,j}^*(t_1)\mathcal{E}_y(F_\beta(t_1)y_t), \quad i + j > 0. \end{aligned}$$

Note that the required quadratic variations are given by $[x]_t = I_x(0, t)$ and $[y]_t = I_y(0, t)$. We further define

$$h_i(x, t) := h_i(x, t, t), \quad i \geq 0, \tag{41}$$

$$g_{i,j}(y, t) := g_{i,j}(y, t, t), \quad i, j \geq 0. \tag{42}$$

Asymptotic Expansion

We propose that a formal asymptotic expansion of $\mathcal{U}(t; v)$ is then possible as

$$\mathcal{U}(t; v) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \epsilon_r^i \epsilon_\lambda^j \mathcal{U}_{i,j}(t; v) \tag{43}$$

where each of the $\mathcal{U}_{i,j}(.)$ is $\mathcal{O}(1)$. By a process of induction we infer that the following recurrence relation must hold:

$$\mathcal{U}_{i,j}(t; v) = -\int_t^v \mathcal{U}_{0,0}(t; u)\left(h(x, u)\mathcal{U}_{i-1,j}(t; u)\mathbb{1}_{i>0} + g(y, u)\mathcal{U}_{i,j-1}(u; v)\mathbb{1}_{j>0}\right) du, \quad i, j \geq 0. \tag{44}$$

Substituting this expression into (43) and and grouping like powers of ϵ_r and ϵ_λ, we can write

$$\mathcal{U}(t; v) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \epsilon_r^i \epsilon_\lambda^j, \mathcal{U}_{i,j}^*(t; v),$$

where in particular we have, following Kato (1995),

$$
\begin{aligned}
\mathcal{U}^*_{0,0}(t;v) &= \mathcal{U}_{0,0}(t;v),\\
\mathcal{U}^*_{1,0}(t;v) &= -\int_t^v \mathcal{U}_{0,0}(t;u)(h_0(x,u)-\tilde{r}(u))\mathcal{U}_{0,0}(u;v)du,\\
\mathcal{U}^*_{0,1}(t;v) &= -\int_t^v \mathcal{U}_{0,0}(t;u)(g_{0,0}(y,u)-\tilde{\lambda}(u))\mathcal{U}_{0,0}(u;v)du,\\
\mathcal{U}^*_{2,0}(t;v) &= -\int_t^v \mathcal{U}_{0,0}(t;u)\left((h_0(x,u)-\tilde{r}(u))\mathcal{U}_{1,0}(u;v)+(h_1(x,u)+g_{1,0}(y,u))\mathcal{U}_{0,0}(u;v)\right)du,\\
\mathcal{U}^*_{0,2}(t;v) &= -\int_t^v \mathcal{U}_{0,0}(t;u)\Big((g_{0,0}(y,u)-\tilde{\lambda}(u))\mathcal{U}_{0,1}(u;v)+g_{0,1}(y,u)\mathcal{U}_{0,0}(u;v)\Big)du,\\
\mathcal{U}^*_{1,1}(t;v) &= -\int_t^v \mathcal{U}_{0,0}(t;u)\Big((h_0(x,u)-\tilde{r}(u))\mathcal{U}_{0,1}(u;v)+(g_{0,0}(y,u)-\tilde{\lambda}(u))\mathcal{U}_{1,0}(u;v)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + g_{1,0}(y,u)\mathcal{U}_{0,0}(u;v)\Big)du.
\end{aligned}
$$

Taking $G_{i,j}(.)$ to be the integral kernel of $\mathcal{U}^*_{i,j}$, we deduce:

$$
G^*(x,y,t;\xi,\eta,v) = B(t,v)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\epsilon_r^i\epsilon_\lambda^j G_{i,j}(x,y,t;\xi,\eta,v). \tag{45}
$$

Following the approach taken by Pagliarani et al. (2011), we seek to obtain explicitly the form of the required integral kernels. At first order we obtain, after some manipulations:

$$
\begin{aligned}
G_{1,0}(x,y,t;\xi,\eta,v) &= -\int_t^v\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{0,0}(x,y,t;x_1,y_1,t_1)(h_0(x_1,t_1)-\tilde{r}(t_1))\\
&\qquad\qquad\qquad\qquad\qquad G_{0,0}(x_1,y_1,t_1;\xi,\eta,v)\,dx_1dy_1dt_1\\
&= -\int_t^v (h_0(x,t,t_1)\mathcal{M}^x_{t,t_1}-\tilde{r}(t_1))G_{0,0}\left(x,y,t;\xi,\eta,v\right)dt_1
\end{aligned} \tag{46}
$$

and

$$
\begin{aligned}
G_{0,1}(x,y,t;\xi,\eta,v) &= -\int_t^v\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{0,0}(x,y,t;x_1,y_1,t_1)(g_{0,0}(y_1,t_1)-\tilde{\lambda}(t_1))\\
&\qquad\qquad\qquad\qquad\qquad G_{0,0}(x_1,y_1,t_1;\xi,\eta,v)\,dx_1dy_1dt_1\\
&= -\int_t^v (g_{0,0}(y,t,t_1)\mathcal{M}^y_{t,t_1}-\tilde{\lambda}(t_1))G_{0,0}(x,y,t;\xi,\eta,v)\,dt_1,
\end{aligned} \tag{47}
$$

where we have defined the shift operators

$$
\mathcal{M}^x_{t_1,t_2}G_{0,0}(x,y,t;\xi,\eta,v) := G_{0,0}\left(x,y,t;\xi-F_\beta(t_2)I_x(t_1,t_2),\eta,v\right) \tag{48}
$$

$$
\mathcal{M}^y_{t_1,t_2}G_{0,0}(x,y,t;\xi,\eta,v) := G_{0,0}\left(x,y,t;\xi,\eta-e^{-\alpha_\lambda t_2}I_y(t_1,t_2),v\right) \tag{49}
$$

Similarly at second order we obtain

$$
\begin{aligned}
G_{2,0}(x,y,t;\xi,\eta,v) &= \int_t^v\int_t^{t_2}(h_0(x,t,t_1)\mathcal{M}^x_{t,t_1}-\tilde{r}(t_1))(h_0(x,t,t_2)\mathcal{M}^x_{t_1,t_2}-\tilde{r}(t_2))\\
&\qquad\qquad\qquad\qquad\qquad G_{0,0}(x,y,t;\xi,\eta,v)\,dt_1dt_2\\
&\quad+\int_t^v(h_0(x,t,t_2)\int_t^{t_2}(h_0(x,t,t_1)\left(\exp\left(F_\beta(t_1)F_\beta(t_2)I_x(0,t_1)\right)-1\right)\\
&\qquad\qquad\qquad\qquad\qquad \mathcal{M}^x_{t,t_1}\mathcal{M}^x_{t_1,t_2}G_{0,0}(x,y,t;\xi,\eta,v)\,dt_1dt_2\\
&\quad-\int_t^v(h_1(x,t,t_1)\mathcal{M}^x_{t,t_1}+g_{1,0}(y,t,t_1)\mathcal{M}^y_{t,t_1})G_{0,0}\left(x,y,t;\xi,\eta,v\right)dt_1,
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
G_{0,2}(x,y,t;\xi,\eta,v) = {} & \int_t^v \int_t^{t_2} (g_{0,0}(y,t,t_1)\mathcal{M}^y_{t,t_1} - \tilde{\lambda}(t_1))(g_{0,0}(y,t,t_2)\mathcal{M}^y_{t_1,t_2} - \tilde{\lambda}(t_2)) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad G_{0,0}(x,y,t;\xi,\eta,v)\, dt_1 dt_2 \\
& + \int_t^v (g_{0,0}(y,t,t_2) \int_t^{t_2} (g_{0,0}(y,t,t_1) \left(\exp\left(e^{-\alpha_\lambda(t_1+t_2)} I_y(0,t_1)\right) - 1\right) \\
& \qquad\qquad\qquad\qquad\qquad \mathcal{M}^y_{t,t_1}\mathcal{M}^y_{t_1,t_2} G_{0,0}(x,y,t;\xi,\eta,v)\, dt_1 dt_2 \\
& - \int_t^v g_{0,1}(y,t,t_1)\mathcal{M}^y_{t,t_1} G_{0,0}(x,y,t;\xi,\eta,v)\, dt_1
\end{aligned} \tag{51}
$$

and

$$
\begin{aligned}
G_{1,1}(x,y,t;\xi,\eta,v) = {} & \int_t^v \int_t^{t_2} (h_0(x,t,t_1)\mathcal{M}^x_{t,t_1} - \tilde{r}(t_1))(g_{0,0}(y,t,t_2)\mathcal{M}^y_{t_1,t_2} - \tilde{\lambda}(t_2)) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad G_{0,0}(x,y,t;\xi,\eta,v)\, dt_1 dt_2 \\
& + \int_t^v \int_t^{t_2} (g_{0,0}(y,t,t_1)\mathcal{M}^y_{t,t_1} - \tilde{\lambda}(t_1))(h_0(x,t,t_2)\mathcal{M}^x_{t_1,t_2} - \tilde{r}(t_2)) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad G_{0,0}(x,y,t;\xi,\eta,v)\, dt_1 dt_2 \\
& + \int_t^v g_{0,0}(x,t,t_2) \int_t^{t_2} h_0(x,t,t_1) \left(\exp\left(e^{-\alpha_\lambda t_2} F_\beta(t_1) I_\rho(0,t_1)\right) - 1\right) \\
& \qquad\qquad\qquad\qquad\qquad \mathcal{M}^x_{t,t_1}\mathcal{M}^y_{t_1,t_2} G_{0,0}(x,y,t;\xi,\eta,v)\, dt_1 dt_2 \\
& + \int_t^v h_0(x,t,t_2) \int_t^{t_2} g_{0,0}(y,t,t_1) \left(\exp\left(e^{-\alpha_\lambda t_1} F_\beta(t_2) I_\rho(0,t_1)\right) - 1\right) \\
& \qquad\qquad\qquad\qquad\qquad \mathcal{M}^y_{t,t_1}\mathcal{M}^x_{t_1,t_2} G_{0,0}(x,y,t;\xi,\eta,v)\, dt_1 dt_2 \\
& - \int_t^v g_{1,0}(y,t,t_1)\mathcal{M}^y_{t,t_1} G_{0,0}(x,y,t;\xi,\eta,v)\, dt_1.
\end{aligned} \tag{52}
$$

Substituting (46)–(52) into (45) gives the required Green's function to second order accuracy.

4 Calibration

It remains to calibrate our model consistent with the no-arbitrage conditions (5) and (6). This is achieved by considering in the former case the consistent pricing of a risk-free cash flow, and in the latter case by going on to consider a risky cash flow, as we shall now show.

Pricing of risk-free cash flow

The calculation for a risk-free cash flow in our model is very similar to that performed by Horvath et al. (2017) and essentially corresponds to taking the distinguished limit as $\epsilon_\lambda \to 0$ then $\epsilon_r \to 0$. The same result is naturally obtained, namely that $f_t^T = X^T(x,t)$ where

$$
X^T(x,t) \sim D(t,T)\left(1 - \epsilon_r F_{1,0}(x,t) + \epsilon_r^2 F_{2,0}(x,t)\right) \tag{53}
$$

with $\mathcal{O}(\epsilon_r^3)$ errors, and our Green's function gives rise to

$$
\begin{aligned}
F_{1,0}(x,t) &= \int_t^T (h_0(x,t,t_1) - \tilde{r}(t_1))\, dt_1 \\
F_{2,0}(x,t) &= \tfrac{1}{2}F_{1,0}^2(x,t) - \int_t^T h_1(x,t,t_1)\, dt_1 \\
&\quad + \int_t^T h_0(x,t,t_2) \int_t^{t_2} h_0(x,t,t_1)\left(\exp\left(F_\beta(t_1)F_\beta(t_2)I_x(0,t_1)\right) - 1\right) dt_1 dt_2.
\end{aligned}
$$

Of interest to us here is the conclusion that, setting $x = y = t = 0$ in (53), satisfying (5) above requires us to choose

$$r_0^*(t) = 0, \tag{54}$$

$$r_1^*(t) = \tilde{r}(t) \int_0^t \tilde{r}(u) \left(\exp\left(F_\beta(u) F_\beta(t) I_x(0,u)\right) - 1\right) du, \tag{55}$$

Pricing of risky cash flow

We continue by writing the price at time t of a risky (zero recovery) cash flow at time T as $f_t^T = Y^T(x_t, y_t, t)$, noting that, in this case, $P(x,y) = 1$ and $f_0^T = Y^T(0,0,0) = B(0,T)$. We look to derive the general functional form of $Y^T(.)$ implied by our model, and in the process to determine the conditions on $\lambda^*(t)$ necessary to satisfy (6) above. Applying our asymptotic Green's function to this problem we conclude

$$Y^T(x,y,t) \sim B(t,T)\left(1 - \epsilon_r F_{1,0}(x,t) - \epsilon_\lambda F_{0,1}(y,t) + \epsilon_r^2 F_{2,0}(x,t) + \epsilon_r \epsilon_\lambda F_{1,1}(x,y,t) + \epsilon_\lambda^2 F_{0,2}(y,t)\right) \tag{56}$$

with $\mathcal{O}(\epsilon_r^3 + \epsilon_\lambda^3)$ error, where the $F_{i,0}(x,t)$ are as defined above and

$$\begin{aligned}
F_{0,1}(y,t) &:= \int_t^T \left(g_{0,0}(y,t,t_1) - \tilde{\lambda}(t_1)\right) dt_1, \\
F_{0,2}(y,t) &:= \tfrac{1}{2} F_{0,1}^2(y,t) - \int_t^T g_{0,1}(y,t,t_1)\, dt_1 \\
&\quad + \int_t^T g_{0,0}(y,t,t_2) \int_t^{t_2} g_{0,0}(y,t,t_1) \left(\exp\left(e^{-\alpha_\lambda(t_1+t_2)} I_y(0,t_1)\right) - 1\right) dt_1\, dt_2, \\
F_{1,1}(x,y,t) &:= F_{1,0}(x,t) F_{0,1}(y,t) - \int_t^T g_{1,0}(y,t,t_1)\, dt_1 \\
&\quad + \int_t^T h_0(x,t,t_2) \int_t^{t_2} g_{0,0}(y,t,t_1) \left(\exp\left(e^{-\alpha_\lambda t_1} F_\beta(t_2) I_\rho(0,t_1)\right) - 1\right) dt_1\, dt_2, \\
&\quad + \int_t^T g_{0,0}(y,t,t_2) \int_t^{t_2} h_0(x,t,t_1) \left(\exp\left(e^{-\alpha_\lambda t_2} F_\beta(t_1) I_\rho(0,t_1)\right) - 1\right) dt_1\, dt_2.
\end{aligned}$$

Setting $x = y = t = 0$, we find that the no-arbitrage condition $Y^T(0,0,0) = B(0,T)$ is satisfied by the expression in (56) iff we choose

$$\lambda_{0,0}^*(t) = 0, \tag{57}$$

$$\lambda_{0,1}^*(t) = \tilde{\lambda}(t) \int_0^t \tilde{\lambda}(u) \left(\exp\left(e^{-\alpha_\lambda(t+u)} I_y(0,u)\right) - 1\right) du. \tag{58}$$

$$\begin{aligned}
\lambda_{1,0}^*(t) &= \tilde{r}(t) \int_0^t \tilde{\lambda}(u) \left(\exp\left(e^{-\alpha_\lambda u} F_\beta(t) I_\rho(0,u)\right) - 1\right) du \\
&\quad + \tilde{\lambda}(t) \int_0^t \tilde{r}(u) \left(\exp\left(e^{-\alpha_\lambda t} F_\beta(u) I_\rho(0,u)\right) - 1\right) du.
\end{aligned} \tag{59}$$

This completes the calibration of our model to second order.

5 Pricing of Protection Leg

We next extend our model to the pricing of a protection leg, viz. an instrument which pays $1 - R$ times the trade notional upon default of the named debt at some stopping time τ^*, zero otherwise, where R is the recovery level on the debt (assumed known in advance as is customary in CDS markets). Let us suppose

now that the process $f_t^T = \hat{p}(\hat{x}_t, \hat{y}_t, t)$ in our original unscaled notation represents the price per unit notional of an instrument offering protection until some future time T. This will satisfy:

$$\left(\frac{\partial}{\partial t} + \hat{\mathcal{L}} - r(\hat{x}, t) - \lambda(\hat{x}, t)\right) \hat{p}(\hat{x}_t, \hat{y}_t, t) = -(1-R)\lambda(\hat{y}, t). \tag{60}$$

subject to the final condition $\hat{p}(\hat{x}, \hat{y}, T) = 0$. Switching to scaled notation and defining a function

$$p(x, y, t) := \epsilon_\lambda (1-R)^{-1} f_t^T|_{x_t = x, y_t = y},$$

this can be seen to satisfy:

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - \epsilon_r h(x, t) - \epsilon_\lambda g(y, t)\right) p(x, y, t) = -(\tilde{\lambda}(t) + g(y, t)) \tag{61}$$

subject to the final condition $p(x, y, T) = 0$. As is evident, this is nothing other than a nonhomogeneous version of (21), hence can be solved (asymptotically) using the Green's function $G^*(\cdot)$ defined previously. Truncating the Green's function in this case at first order and applying it in the standard manner, we deduce:

$$p(x, y, t) \sim P_{0,0}(y, t) + \epsilon_\lambda P_{0,1}(y, t) + \epsilon_r P_{1,0}(x, y, t) \tag{62}$$

with $\mathcal{O}(\epsilon_r^2 + \epsilon_\lambda^2)$ error, where

$$\begin{aligned} P_{0,0}(y, t) &= \int_t^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(t, t_1) G_{0,0}(x, y, t; \xi, \eta, t_1) g_{0,0}(\eta, t_1)\, d\xi\, d\eta\, dt_1 \\ &= \int_t^T B(t, t_1) g_{0,0}(y, t, t_1)\, dt_1, \end{aligned} \tag{63}$$

$$\begin{aligned} P_{0,1}(y, t) &= \int_t^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(t, t_2) G_{0,1}(x, y, t; \xi, \eta, t_2) g_{0,0}(\eta, t_2)\, d\xi\, d\eta\, dt_2 \\ &\quad + \int_t^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(t, t_2) G_{0,0}(x, y, t; \xi, \eta, t_2) g_{0,1}(\eta, t_2)\, d\xi\, d\eta\, dt_2 \\ &= -\int_t^T B(t, t_2) g_{0,0}(y, t, t_2) \int_t^{t_2} g_{0,0}(y, t, t_1) \left(\exp\left(e^{-\alpha_\lambda (t_1 + t_2)} I_y(t, t_1)\right) - 1\right) dt_1\, dt_2 \\ &\quad + \int_t^T B(t, t_1) g_{0,1}(y, t, t_1)\, dt_1 \end{aligned} \tag{64}$$

and

$$\begin{aligned} P_{1,0}(x, y, t) &= \int_t^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(t, t_2) G_{1,0}(x, y, t; \xi, \eta, t_2) g_{0,0}(\eta, t_2)\, d\xi\, d\eta\, dt_2 \\ &\quad + \int_t^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(t, t_2) G_{0,0}(x, y, t; \xi, \eta, t_2) g_{1,0}(\eta, t_2)\, d\xi\, d\eta\, dt_2 \\ &= -\int_t^T B(t, t_2) g_{0,0}(y, t, t_2) \int_t^{t_2} h_0(x, t, t_1) \left(\exp\left(F_\beta(t_1) e^{-\alpha_\lambda t_2} I_\rho(t, t_1)\right) - 1\right) dt_1\, dt_2 \qquad (65) \\ &\quad + \int_t^T B(t, t_1) g_{1,0}(y, t, t_1)\, dt_1. \qquad (66) \end{aligned}$$

Finally, noting that $P_{0,1}(0,0) = 0$ and combining the relevant contributions from $P_{0,0}(0,0)$ and $P_{1,0}(0,0,0)$, we conclude in terms of our original (unscaled) notation that the PV of the protection leg is given asymptotically by

$$PV_{t=0} \sim (1-R) \int_0^T B(0, u) \left(\overline{\lambda}(u) \left(1 - \phi_P(u)\right) + \Delta\lambda_1(u)\right) du \tag{67}$$

with $\mathcal{O}(\epsilon_\lambda(\epsilon_r^2+\epsilon_\lambda^2))$ error, where

$$\phi_P(u) := \int_0^u \frac{\overline{r}(v)}{1-\beta}\left(\exp\frac{(1-\beta)e^{-\alpha_r(u-v)}I_R(0,v)}{|\overline{r}(v)|^\beta}-1\right)dv, \tag{68}$$

$$I_R(t_1,t_2) := \rho_{r\lambda}\int_{t_1}^{t_2} e^{-(\alpha_r+\alpha_\lambda)(t_2-u)}\sigma_r(u)\sigma_\lambda(u))du, \tag{69}$$

$$\Delta\lambda_1(u) := \int_0^u \phi_{1,1}(u,v)\,dv, \tag{70}$$

$$\begin{aligned}\phi_{1,1}(u,v) := {} & \frac{\overline{r}(u)\overline{\lambda}(v)}{1-\beta}\left(\exp\frac{(1-\beta)e^{-\alpha_r(u-v)}I_R(0,v)}{|\overline{r}(u)|^\beta}-1\right)\\ & + \frac{\overline{\lambda}(u)\overline{r}(v)}{1-\beta}\left(\exp\frac{(1-\beta)e^{-\alpha_\lambda(u-v)}I_R(0,v)}{|\overline{r}(v)|^\beta}-1\right).\end{aligned} \tag{71}$$

Note that $\Delta\lambda_1(t)$ here is none other than $\lambda^*_{1,0}(t)$, expressed in unscaled notation.

As expected, cancellation results in there being no contribution at $\mathcal{O}(\epsilon_\lambda^2)$. Further, in the absence of correlation, $\phi_P(\cdot) = \Delta\lambda_1(\cdot) = 0$ and the value of protection is as given under the assumption of deterministic rates. We further note that it is a straightforward matter to take the limit as $\beta \uparrow 1$ in (68) and (71) to obtain the results for the case of a Hull-White interest rate model. In particular, we find

$$\phi_P(u) \to \int_0^u e^{-\alpha_r(u-v)}I_R(0,v)\,dv$$

$$\Delta\lambda_1(u) \to \int_0^u \left(\overline{\lambda}(v)e^{-\alpha_r(u-v)} + \overline{\lambda}(u)e^{-\alpha_\lambda(u-v)}\right)I_R(0,v)\,dv$$

6 Pricing of Fixed Coupon Leg

For a payment period (t_{i-1},t_i), $i \geq 1$, the payoff for a fixed coupon flow will be given by $c\delta(t_{i-1},t_i)$, with $\delta(\cdot)$ the year fraction under the relevant day count convention. The PV of this (defaultable) cash flow is given straightforwardly by discounting it with the relevant risky discount factor, namely $B(0,t_i)$. If a default occurs at a time $\tau^* \in (t_{i-1},t_i)$, there may be a payment at time τ^* of the accrued interest amount, namely

$$\text{Accrual} = c\delta(t_{i-1},\tau^*),$$

if such is contractually specified.

We look to estimate the value of such payments under our model. Our task is simplified by realising that the value of accrual is essentially a forward-starting protection payment with a time-dependent payoff. The calculation is virtually identical to that in §5 above with the accrual amount incorporated into the r.h.s. of (61). We omit the details and simply state the result that, taking account of accrual, the value of the cash flow(s) associated with the ith payment period is given for $t_i > 0$ by

$$PV_{t=0}^{(i)} \sim cB(0,t_i)\delta(t_{i-1},t_i) + c\int_{0\vee t_{i-1}}^{t_i} B(0,u)\delta(t_{i-1},u)\left(\overline{\lambda}(u)\left(1-\phi_P(u)\right)+\Delta\lambda_1(u)\right)du \tag{72}$$

with $\mathcal{O}(\epsilon_\lambda(\epsilon_r^2+\epsilon_\lambda^2))$ error, and zero otherwise. Technically, since the coupon c should be considered to be $\mathcal{O}(\epsilon_r)$, the adjustment term here is, through the inclusion of $\phi_P(u)$, an order of approximation smaller than those already considered; however, we include it here for completeness.

With the above results and those of the previous section, we are in a position to infer prices for credit default swaps (CDS) for a given assumed term structure of zero coupon bond prices, or alternatively to calibrate our model to a given term structure of CDS rates, as we shall now show.

7 Calibration to CDS Market

In practice, our model would often be calibrated to a term structure of CDS prices rather than to zero coupon bond prices. We therefore seek to demonstrate how this could be done making use of the methodology developed in the preceding section. First of all, let us define the calibration CDS to have maturities given by T_j, $j = 1, 2, \ldots, n$ with corresponding par rates given by s_j. Let us further suppose that the function $\overline{\lambda}(t)$ can be taken as piecewise constant between the T_j, given say by

$$\overline{\lambda}(t) = \lambda_j, \quad t \in (T_{j-1}, T_j].$$

We can write the PV of the coupon leg of the jth CDS with coupon amount c straightforwardly using the risky discount factors as[4]

$$F_j(c) = c \sum_{i=1}^{N_j} B(0, t_i)\delta(t_{i-1}, t_i)\mathbb{1}_{t_i > 0} \tag{73}$$

where t_0 is the start date, the payment dates are $t_1, t_2, \ldots, t_n$, $t_{N_j} = T_j$ and $\delta(.)$ specifies the relevant year fraction for the payment period. Denoting the PV of the respective protection legs by P_j, it is convenient to define the incremental protection values

$$\begin{aligned} \Delta P_j &:= P_j - P_{j-1} \\ &\sim (1-R)\lambda_j \int_{T_{j-1}}^{T_j} B(0,u)\,(1 - \phi_P(u))\,du \end{aligned} \tag{74}$$

where, for a given estimate of λ_j, there will be a weak dependence of $B(0, u)$ on that estimate, but no such dependence for P_{j-1} and we take $P_0 \equiv 0$. Successful calibration for the jth instrument requires

$$\Delta P_j = F_j(s_j) - F_{j-1}(s_{j-1}),$$

with $F_0 \equiv 0$. These equations can easily be solved in turn iteratively by making an initial estimate, say $\lambda_1^{(0)} = s_1$, or $\lambda_j^{(0)} = \lambda_{j-1}$ for $j > 1$, computing $F_j^{(0)}(s_j)$ and $\Delta P_j^{(0)}$ based on the implied $B(0, u)$ for $T_{j-1} \le u \le T_j$ (the $F_{j-1}(s_{j-1})$ term being already known from the previous calibration step) and using as the next estimate

$$\lambda_j^{(1)} = \frac{\Delta F_j^{(0)}(s_j)}{\Delta P_j^{(0)}} \lambda_j^{(0)}$$

and so on until a converged value for λ_j emerges, for each value of j in turn.

8 Defaultable Libor Flow

Finally we give consideration to the pricing of defaultable Libor payments. This capability will be required in circumstances where a CDS is issued which pays floating rate coupons. This is particularly common for credit linked notes where, unlike with a CDS, an upfront notional is invested with protection sold thereon against a named debt issuer: typically Libor payments are made against the invested notional, plus a spread to take account of the credit risk. We consider a tenor-τ credit-contingent Libor flow paying at time T. This has zero value for $t \ge T$ and can be treated as a fixed flow if $t \in [T - \tau, T)$ and the Libor rate has already fixed. Hence we focus our attention on the case $t < T - \tau$.

[4] We assume here that there is no accrued coupon paid at default. The method to take into account the PV impact of contractual accrued coupon payments is described in §6 above.

Survival-contingent payment

The payoff at time T of a tenor-τ credit-contingent Libor flow is given in terms of our previously defined notation by

$$\begin{aligned}\text{Libor} &= X^T(x_{T-\tau}, T-\tau)^{-1} - 1 \\ &= \epsilon_r F_0 + D(T-\tau,T)^{-1}\left(\epsilon_r F_{1,0}(x_{T-\tau}, T-\tau) + \epsilon_r^2 (F_{1,0}^2(x_{T-\tau}, T-\tau) - F_{2,0}(x_{T-\tau}, T-\tau))\right) + O(\epsilon_r^3),\end{aligned}$$

where

$$F_0 := (1-\beta)\int_{T-\tau}^{T} \tilde{r}(t_1)\, dt_1.$$

This final condition at time T requires specification of the unknown value of the stochastic variable $x_{T-\tau}$. We proceed by using our (first order) Green's function first to value this payoff as of the fixing time $t = T-\tau$, at which $x_{T-\tau}$ can be assumed known. The payoff can therefore be considered as a fixed (risky) cash flow and evaluated as such. Denoting the time-t value of the Libor flow by $F_L(x_t, y_t, t)$ and expanding as

$$F_L(x,y,t) = \epsilon_r H_{1,0}(x,t) + \epsilon_r^2 H_{2,0}(x,t) + \epsilon_r \epsilon_\lambda H_{1,1}(x,y,t) + \mathcal{O}(\epsilon_r(\epsilon_r^2 + \epsilon_\lambda^2)), \tag{75}$$

we obtain straightforwardly

$$\begin{aligned} H_{1,0}(x, T-\tau) &= B(T-\tau,T)(F_0 + D(T-\tau,T)^{-1} F_{1,0}(x, T-\tau)), \\ H_{2,0}(x, T-\tau) &= \frac{B(T-\tau,T)}{D(T-\tau,T)}(F_{1,0}^2(x,T-\tau) - F_{2,0}(x,T-\tau)) \\ &\quad - H_{1,0}(x,y,T-\tau)\int_{T-\tau}^{T}(h_0(x,T-\tau,t_1) - \tilde{r}(t_1))\, dt_1 \\ H_{1,1}(x,y,T-\tau) &= -H_{1,0}(x,y,T-\tau)\int_{T-\tau}^{T}(g_{0,0}(y,T-\tau,t_1) - \tilde{\lambda}(t_1))\, dt_1 \end{aligned}$$

Taking this value as a payoff at time $T-\tau$ and applying our leading order Green's function for $t < T-\tau$, we obtain from $H_{1,0}(x, T-\tau)$:

$$H_{1,0}(x,t) = B(t,T)\left(F_0 + D(T-\tau,T)^{-1}\int_{T-\tau}^{T}(h_0(x,t,t_1) - \tilde{r}(t_1))\, dt_1\right) \tag{76}$$

and consequently $H_{1,0}(0,0) = B(0,T)F_0$. In obtaining $H_{1,1}(x,y,t)$, we observe that contributions will arise from applying $G_{0,1}(\cdot)$ to $H_{1,0}(\cdot)$ *and* from applying $G_{0,0}(\cdot)$ to $H_{1,1}(\cdot)$. We obtain:

$$
\begin{aligned}
H_{1,1}(x,y,t) &= -B(t,T-\tau)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{0,1}(x,y,t;\xi,\eta,T-\tau)H_{1,0}(\xi,T-\tau)\,d\xi\,d\eta \\
&\quad + B(t,T-\tau)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{0,0}(x,y,t;\xi,\eta,T-\tau)H_{1,1}(\xi,\eta,T-\tau)\,d\xi\,d\eta \\
&= -B(t,T)\int_t^{T-\tau}(g_{0,0}(y,t,t_2)-\tilde{\lambda}(t_2))\,dt_2 \\
&\qquad\qquad \left(F_0 + D(T-\tau,T)^{-1}\int_{T-\tau}^{T}(h_0(x,t,t_1)-\tilde{r}(t_1))\,dt_1\right) \\
&\quad - \frac{B(t,T)}{D(T-\tau,T)}\int_t^{T-\tau} g_{0,0}(y,t,t_2)\int_{T-\tau}^{T} h_0(x,t,t_1)\left(\exp\left(e^{-\alpha_\lambda t_2}F_\beta(t_1)I_\rho(0,t_2)\right)-1\right)dt_1dt_2 \\
&\quad - B(t,T)\left(F_0 + D(T-\tau,T)^{-1}\int_{T-\tau}^{T}(h_0(x,t,t_2)-\tilde{r}(t_2))\,dt_2\right)\int_{T-\tau}^{T}(g_{0,0}(y,t,t_1)-\tilde{\lambda}(t_1))\,dt_1 \\
&\quad - \frac{B(t,T)}{D(T-\tau,T)}\int_{T-\tau}^{T} h_0(x,t,t_2)\int_{T-\tau}^{T} g_{0,0}(y,t,t_1) \\
&\qquad\qquad \left(\exp\left(e^{-\alpha_\lambda t_1}F_\beta(t_2)I_\rho(0,t_1\wedge t_2)\right)-1\right)dt_1dt_2
\end{aligned} \tag{77}
$$

A similar expression can be derived for $H_{2,0}(x,t)$ by applying $G_{1,0}(\cdot)$ to $H_{1,0}(\cdot)$ and $G_{0,0}(\cdot)$ to $H_{2,0}(\cdot)$ but, since in the absence of any ϵ_λ dependence it necessarily gives zero contribution at $t=0$, we do not produce it explicitly here. Substituting (76) and (77) back into (75), setting $x=y=t=0$ and reverting to unscaled notation, we see that our defaultable Libor flow has PV given by

$$
PV_{t=0} \sim B(0,T)\left(\frac{1-\phi_L}{D(T-\tau,T)}-1\right), \tag{78}
$$

with $\mathcal{O}(\epsilon_r(\epsilon_r^2+\epsilon_\lambda^2))$ error, where

$$
\phi_L := \int_0^T \overline{\lambda}(v)\int_{T-\tau}^{T}\frac{\overline{r}(u)}{1-\beta}\left(\exp\frac{(1-\beta)\gamma(u,v)I_R(0,u\wedge v)}{|\overline{r}(u)|^\beta}-1\right)du\,dv, \tag{79}
$$

$$
\gamma(u,v) := \begin{cases} e^{-\alpha_\lambda(v-u)}, & u\le v, \\ e^{-\alpha_r(u-v)}, & u>v. \end{cases} \tag{80}
$$

Setting $\phi_L = 0$, equivalent to zero correlation, recovers the price under the assumption of deterministic rates. As expected, negative correlation increases the value of the Libor flow (with risky discount factors $B(0,t)$ assumed fixed) and *vice versa*. Also, taking the Hull-White limit as $\beta\uparrow 1$, we find

$$
\phi_L \to \int_0^T \overline{\lambda}(v)\int_{T-\tau}^{T}\gamma(u,v)I_R(0,u\wedge v)\,du\,dv \tag{81}
$$

We note that, provided $\overline{\lambda}(\cdot)$ remains bounded and $\alpha_r,\alpha_\lambda>0$, the value of ϕ_L in (81) remains bounded in the limit as $T\to\infty$ with τ fixed, whence we conclude that the derived Hull-White formula is uniformly asymptotic in that limit. This will not be the case if either of the mean reversion rates are zero. Under the assumption that $\overline{r}(\cdot)$ also remains bounded, the same is seen to be true of (79).

Adjustment for accrual

For completeness we consider also the impact of any accrual payment which becomes due upon default. We focus on the accrual associated with a payment period (t_{i-1},t_i). The case $t_{i-1}\ge 0$ can again be treated as

a fixed flow so we suppose w.l.o.g. that $t_{i-1} > 0$. The payoff associated with default at a time $\tau^* \in (t_{i-1}, t_i)$ of a credit-contingent Libor leg is given in terms of our previously defined notation by

$$\begin{aligned}\text{Accrual} &= \Delta_i(\tau^*)(X^{t_i}(x_{t_{i-1}}, t_{i-1})^{-1} - 1)\\ &= \epsilon_r \Delta_i(\tau^*)\left(F_0 + D(t_{i-1}, t_i)^{-1} F_{1,0}(x_{t_{i-1}}, t_{i-1})\right) + O(\epsilon_r^2),\end{aligned}$$

where

$$\Delta_i(u) := \frac{\delta(t_{i-1}, u)}{\delta(t_{i-1}, t_i)}, \quad u \in [t_{i-1}, t_i], \tag{82}$$

with $\delta(\cdot)$ the year fraction under the relevant day count convention. Writing the PV contribution associated with such a default as $\Delta H^{(i)}(x, y, t)$ for $t < t_i$ and applying the same approach as in §5 above, we find the value as of time $t \in [t_{i-1}, t_i)$ is given by

$$\Delta H^{(i)}(x_{t_{i-1}}, y, t) = \epsilon_r \epsilon_\lambda \Delta H^{(i)}_{1,1}(x_{t_{i-1}}, y, t) + \mathcal{O}(\epsilon_r \epsilon_\lambda(\epsilon_r + \epsilon_\lambda)) \tag{83}$$

where for $t \geq t_{i-1}$

$$\begin{aligned}\Delta H^{(i)}_{1,1}(x, y, t) = \int_t^{t_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} & B(t, v) G_{0,0}(x, y, t; \xi, \eta, v) g_{0,0}(\eta, v) \Delta_i(v)\\ & \left(F_0 + D(t_{i-1}, t_i)^{-1} F_{1,0}(x, t_{i-1})\right) d\xi\, d\eta\, dv,\end{aligned}$$

whence

$$\Delta H^{(i)}_{1,1}(x, y, t_{i-1}) = \int_{t_{i-1}}^{t_i} B(t_{i-1}, v) g_{0,0}(y, t_{i-1}, v) \Delta_i(v) \left(F_0 + D(t_{i-1}, t_i)^{-1} F_{1,0}(x, t_{i-1})\right) dv. \tag{84}$$

Turning our attention to the case $t < t_{i-1}$, we take (84) as specifying the payoff at time t_{i-1}; applying our leading order Green's function we obtain straightforwardly:

$$\begin{aligned}\Delta H^{(i)}_{1,1}(x, y, t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(t, t_{i-1}) G_{0,0}(x, y, t; \xi, \eta, t_{i-1}) \Delta H^{(i)}_{1,1}(\xi, \eta, t_{i-1})\, d\xi\, d\eta\\ &= F_0 \int_{t_{i-1}}^{t_i} B(t, v) g_{0,0}(y, t, v) \Delta_i(v) dv + D(t_{i-1}, t_i)^{-1} \int_{t_{i-1}}^{t_i} B(t, v) g_{0,0}(y, t, v) \Delta_i(v)\\ &\qquad \int_{t_{i-1}}^{t_i} h_0(x, t, u) \left(\exp\left(e^{-\alpha_\lambda v} F_\beta(u) I_\rho(t, u \wedge v)\right) - 1\right) du\, dv, \quad t < t_{i-1}.\end{aligned} \tag{85}$$

Setting $x = y = t = 0$ and reverting to original unscaled notation we obtain

$$\Delta PV^{(i)}_{t=0} \sim \int_{t_{i-1}}^{t_i} B(0, v) \overline{\lambda}(v) \Delta_i(v) \left(\frac{1 + \phi_A(v)}{D(t_{i-1}, t_i)} - 1\right) dv. \tag{86}$$

where[5]

$$\phi_A(v) := \int_{t_{i-1}}^{t_i} \frac{\overline{r}(u)}{1 - \beta} \left(\exp \frac{(1 - \beta)\gamma(u, v) I_R(0, u \wedge v)}{|\overline{r}(u)|^\beta} - 1\right) du. \tag{87}$$

Note that in the Hull-White limit $\beta \uparrow 1$ we have

$$\phi_A(v) \to \int_{t_{i-1}}^{t_i} \gamma(u, v) I_R(0, u \wedge v)\, du. \tag{88}$$

[5]As with the fixed coupon case, the accrual term is on order of approximation smaller than the main Libor contribution, so the $\phi_A(v)$ term here can be consistently ignored and is included only for completeness. However, if it is wished to include the impact of terms at $\mathcal{O}(\epsilon_\lambda \epsilon_r^2)$ in the accrual term, arguably a term $\Delta\lambda_1(v)$ should also be included alongside $\overline{\lambda}(v)$ in (86), capturing the impact of $\lambda^*_{1,0}(\cdot)$.

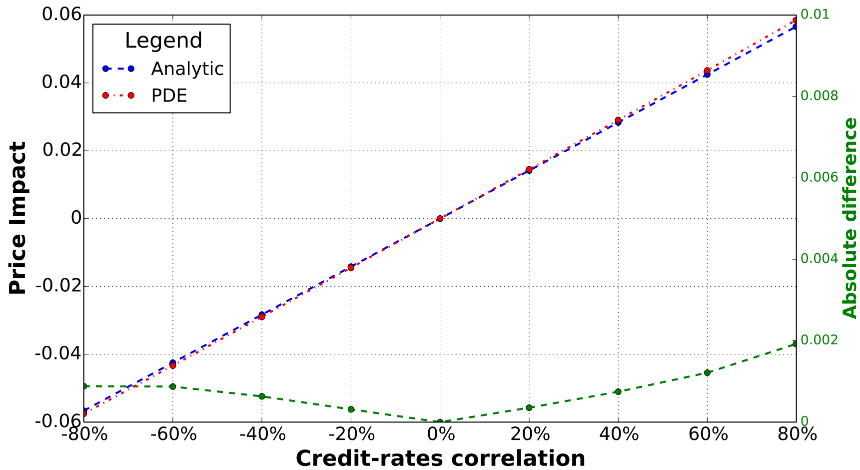

Figure 1: Impact of correlation on PV for a 5y maturity CDS

9 Comparison of Results

A test is made of the accuracy of our CDS pricing formulae in the case of a Hull-White interest rate model. A 5y vanilla CDS was considered with a quarterly ATM coupon of 400 bp on a notional of 100M EUR. It was assumed that accrual was paid upon default. The zero rate was taken to rise from around 1.5% to around 3% over a five-year period. The credit spread was taken to be $\lambda(t) = 6.5\%$ with an assumed recovery level of 40%. The (normal) interest rate volatility $\sigma_r(t)$ was taken to be 0.5% with mean reversion rate $\alpha_r = 0.25$ and the (lognormal) credit volatility $\sigma_\lambda(t)$ was taken to be 60% with mean reversion rate $\alpha_\lambda = 0.3$. Calculations were carried out using our approximate formulae above and results compared with those from a finite difference solution of the relevant PDE. The impact of correlation on PV is illustrated for a range of correlations $\rho_{r\lambda}$ in Fig. 1. As can be seen, even up to a very large correlation of 80%, the differences nowhere exceed 0.002M EUR, viz. 0.2 bp of notional. In a similar calculation for a 10y CDS it was found that the maximum difference increased to 0.8 bp of notional, still of a negligible size.

A similar test was done for an interest swap extinguisher. This is a trade where a Libor leg is exchanged with a fixed coupon leg, with all future cash flows being cancelled if a named issuer defaults. Typically, accrued coupon is paid up to the date of default. We consider the same fixed coupon leg as previously matched with a float leg paying Libor plus 100bp, which leaves the 5y swap quite close to the money. Calculations were again carried out using our approximate formulae above and results compared with those from a finite difference solution of the relevant PDE. The impact of correlation on PV is illustrated for a range of correlations $\rho_{r\lambda}$ in Fig. 2. As can be seen, the agreement is excellent. Notably the impact of correlation on the coupon flows is considerably less that on a protection leg.

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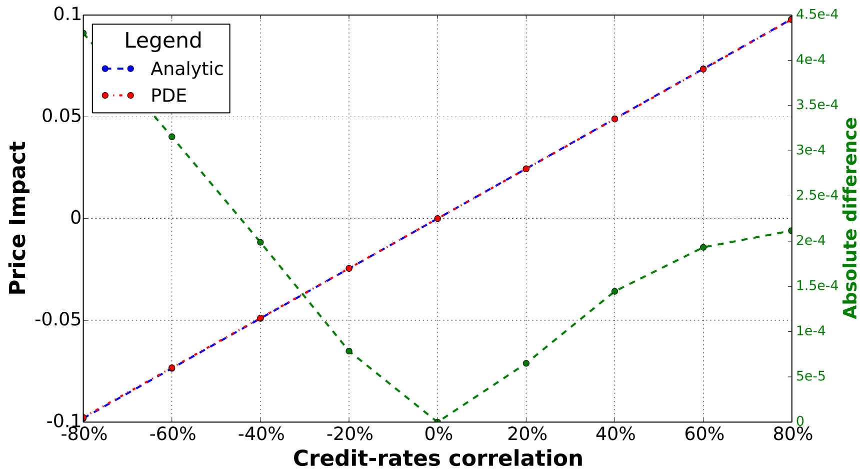

Figure 2: Impact of correlation on PV for a $5y$ maturity interest rate swap extinguisher

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